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The signed and minus k-subdomination numbers of comets

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Abstract

Let $G = (V,E)$ be a graph. For any real valued function $f : V \to \mathbf{R}$ and $S \subseteq V$, let $f(S) = \sum_{u \in S} f(u)$. The weight of f is defined as $f(V)$. A signed k-subdominating function (kSF) of G is defined as a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geqslant 1$ for at least k vertices of G. The signed k-subdomination number of a graph G, denoted by $\gamma_{ks}^{-11}(G)$, is equal to $\min\{f(V) \mid f$ is a signed kSF of $G\}$. A minus kSF and the corresponding parameter, the minus k-subdomination number of G, denoted by $\gamma_{ks}^{-101}(G)$, are defined similarly, except that 0 is now also an allowable value. In this paper we compute the minus and signed k-subdomination numbers for a class of trees called comets.

1. Introduction

Let $G = (V,E)$ be a graph and let v be a vertex in V. The *open neighbourhood* of v is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u \mid uv \in E\}$. The *closed neighbourhood* of v is $N[v] = N(v) \cup \{v\}$. For a set S of vertices, we define the open neighbourhood $N(S)$ as $\bigcup_{v \in S} N(v)$ and the closed neighbourhood $N[S]$ as $N(S) \cup S$. For any real valued function $f : V \to \mathbf{R}$ and $S \subseteq V$, let $f(S) = \sum_{u \in S} f(u)$. The *weight* of f is defined as $f(V)$. We will also denote $f(N[v])$ by $f[v]$, where $v \in V$. We say $v \in V$ is *covered by* f if $f[v] \geqslant 1$. The set of vertices covered by f is denoted by C_f.

A *signed dominating function* is defined in [6] as a function $f : V \to \{-1, 1\}$ such that $|C_f| \geqslant |V|$. The *signed domination number* of a graph G is $\gamma_s(G) = \min\{f(V) \mid f$ is a signed dominating function of $G\}$. A *majority dominating function* is defined in [3] as a function $f : V \to \{-1, 1\}$ such that $|C_f| \geqslant \lceil \frac{|V|}{2} \rceil$. The *majority domination number* of a graph G is $\gamma_{maj}(G) = \min\{f(V) \mid f$ is a majority dominating function of $G\}$. Let k be a positive integer such that $1 \leqslant k \leqslant |V|$. A *signed k-subdominating function* (signed kSF) for G is defined in [4] as a function $f : V \to \{-1, 1\}$ such that $|C_f| \geqslant k$. The *signed k-subdomination number* of a graph G, denoted by $\gamma_{ks}^{-11}(G)$, is equal to $\min\{f(V) \mid f$ is a signed kSF of $G\}$. In the special cases where $k = |V|$ and

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$k = \lceil \frac{|V|}{2} \rceil$, $\gamma_{\mathrm{ks}}^{-11}(G)$ is, respectively, the signed domination number and the majority domination number.

The *comet* $C_{s,t}$, where s and t are positive integers, denotes the tree obtained by identifying the centre of the star $K_{1,s}$ with an end vertex of P_t, the path of order t. So $C_{s,1} \cong K_{1,s}$ and $C_{1,p-1} \cong P_p$. Beineke and Henning (see [1]) computed $\gamma_{\mathrm{ks}}^{-11}(C_{s,t})$ for $k = s+t$ and for $k = \lceil (s+t)/2 \rceil + 1$. This parameter has also been computed for certain values of k for other classes of trees such as full m-ary trees (see [4]). In Section 2 we compute $\gamma_{\mathrm{ks}}^{-11}(C_{s,t})$ for all possible values of k where $1 \leqslant k \leqslant s+t$.

A *minus dominating function* is defined in [5] as a function $f : V \to \{-1,0,1\}$ such that $|C_f| \geqslant |V|$. The *minus domination number* of a graph G is $\gamma^{-}(G) = \min\{f(V) \mid f$ is a minus dominating function of $G\}$. Let k be a positive integer such that $1 \leqslant k \leqslant |V|$. A *minus k-subdominating function* (minus *kSF*) for G is defined in [2] as a function $f : V \to \{-1,0,1\}$ such that $|C_f| \geqslant k$. The *minus k-subdomination number* of a graph G, denoted by $\gamma_{\mathrm{ks}}^{-101}(G)$, is equal to $\min\{f(V) \mid f$ is a minus kSF of $G\}$. In the special case where $k = |V|$, $\gamma_{\mathrm{ks}}^{-101}(G)$ is the minus domination number. In Section 3 we compute $\gamma_{\mathrm{ks}}^{-101}(C_{s,t})$ for all possible values of k where $1 \leqslant k \leqslant s+t$.

2. The signed k-subdomination number of comets

In this section we compute the value of $\gamma_{\mathrm{ks}}^{-11}$ for comets. The following result of Cockayne and Mynhardt [4] will prove to be useful.

Theorem 1. *If T is a tree of order $p \geqslant 2$ and k is an integer such that $1 \leqslant k \leqslant p$, then $\gamma_{\mathrm{ks}}^{-11}(T) \geqslant 2\lfloor (2k+4)/3 \rfloor - p$ with equality for $T = P_p$.*

Theorem 2. *Let p,s and t be positive integers such that $p = s+t$ and let $G = C_{s,t}$. If $s,t \geqslant 2$, then*

$$\gamma_{\mathrm{ks}}^{-11}(G) = \begin{cases} 2\lfloor (2k+4)/3 \rfloor - p & \text{if } k \leqslant t-1, \\ 2(k - \lceil \frac{t}{3} \rceil + 2) - p & \text{if } t \leqslant k \text{ and} (k \leqslant t + \lfloor \frac{s}{2} \rfloor - 2, t \equiv 0 \pmod 3) \text{ or} \\ & \quad k \leqslant t + \lfloor \frac{s}{2} \rfloor, t \equiv 1 \pmod 3 \text{ or} \\ & \quad k \leqslant t + \lfloor \frac{s}{2} \rfloor - 1,\ t \equiv 2 \pmod 3)), \\ 2(k - \lceil \frac{t}{3} \rceil + 1) - p & \text{otherwise.} \end{cases}$$

Proof. Let v be the center of the star, let u be its neighbour on the path and, for $t \geqslant 3$, let $w \in N(u) - \{v\}$.

If $k \leqslant t-1$, then $2\lfloor (2k+4)/3 \rfloor - p$ is a lower bound for $\gamma_{\mathrm{ks}}^{-11}(G)$. For $t = 2$, it is achieved by assigning the value 1 to the vertices u and v and the value -1 to the remaining vertices of G. Then $f(V) = 2 + (p-2)(-1) = 4 - p = 2\lfloor (2 \cdot 1 + 4)/3 \rfloor - p = 2\lfloor (2k+4)/3 \rfloor - p$. For $t \geqslant 3$, it is achieved by taking a minimum signed dominating function on the subpath on k ($\leqslant t-1$) vertices that emanates from the end vertex of the tail of the comet and extending this function to a signed kSF of G by assigning to each remaining vertex the value -1. By Theorem 1, $f(V) = (2\lfloor(2k+4)/3\rfloor - k) - (p-k) = 2\lfloor(2k+4)/3\rfloor - p$.

Now suppose that $k \geqslant t$. Let S denote the set of end vertices of the star. Among all signed kSF's of G achieving $\gamma_{ks}^{-11}(G)$, let f be chosen as to maximise the number of vertices of S that are assigned -1 by f. We show first that $f(v) = 1$. If this is not the case, then $f(v) = -1$. Then $S \cap C_f = \emptyset$ and at most $p - s = t$ vertices, namely the vertices on the path, are in C_f. But then $k \leqslant t$, whence $k = t$ and $v \in C_f$. Since $u \in C_f$, we have $f(u) = f(w) = 1$. Also, since $v \in C_f$, at least $\lfloor \frac{s}{2} \rfloor + 1$ of the vertices in S are assigned the value 1 by f — let v' be one such vertex. Define $g : V \to \{-1, 1\}$ by

$$g(x) = \begin{cases} 1 & \text{if } x = v, \\ -1 & \text{if } x = v', \\ f(x) & \text{otherwise.} \end{cases}$$

Then g is a signed kSF such that $g(V) = f(V)$ while $|\{v \in S \mid g(v) = -1\}| > |\{v \in S \mid f(v) = -1\}|$, contradicting the choice of f. This contradiction shows that $f(v) = 1$.

Before proceeding any further, we prove five claims concerning f.

Claim 3. *If $v \notin C_f$, then $V(P_t) - \{v\} \subseteq C_f$.*

Proof. Since $k \geqslant t$, there is a $v' \in S$ such that $f(v') = 1$. Suppose, to the contrary, that there is a vertex $x \in V(P_t) - \{v\}$ such that $x \notin C_f$. We distinguish between two cases.

Case 1: $f(x) = 1$. If x is an end vertex of P_t, then the vertex adjacent to x is assigned the value -1 by f. If this is not the case, then both vertices adjacent to x are assigned the value -1 by f. Let y be the vertex adjacent to x at shortest distance from v. Then $g : V \to \{-1, 1\}$ defined by

$$g(z) = \begin{cases} 1 & \text{if } z = y, \\ -1 & \text{if } z = v', \\ f(z) & \text{otherwise} \end{cases}$$

is a signed kSF such that $g(V) = f(V)$ while $|\{v \in S \mid g(v) = -1\}| > |\{v \in S \mid f(v) = -1\}|$, contradicting the choice of f.

Case 2: $f(x) = -1$. Let y be the vertex nearest to v for which every vertex on the xy-path is assigned the value -1 by f. Then $g : V \to \{-1, 1\}$ defined by

$$g(z) = \begin{cases} 1 & \text{if } z = y, \\ -1 & \text{if } z = v', \\ f(z) & \text{otherwise} \end{cases}$$

is a signed *kSF* such that $g(V) = f(V)$ while $|\{v \in S \mid g(v) = -1\}| > |\{v \in S \mid f(v) = -1\}|$, contradicting the choice of f.

These contradictions show that $V(P_t) - \{v\} \subseteq C_f$. $\square$

Claim 4. *If $v \notin C_f$, then $f(V) \geqslant 2(k - \lceil \frac{t}{3} \rceil + 2) - p$.*

Proof. First of all, note that $|S \cap C_f| \geqslant k - t + 1$. By Claim 3, $V(P_t) - \{v\} \subseteq C_f$. If $t = 2$, then $f(u) = f(v) = 1$, so that $f(V) \geqslant 2(2 + (k-1)) - p = 2(k - 1 + 2) - p = 2(k - \lceil \frac{t}{3} \rceil + 2) - p$. If $t = 3$, then $f(w) = f(v) = f(u) = 1$, so that $f(V) \geqslant 2(3 + (k-2)) - p = 2(k - 1 + 2) - p = 2(k - \lceil \frac{t}{3} \rceil + 2) - p$.

Now consider the case when $t \geqslant 4$. Let $w' \in N(w) - \{u\}$. We show that we may, without loss of generality, assume that $f(w') = f(w) = f(v) = 1$ and $f(u) = -1$. First suppose that $f(u) = -1$. Then, $u \in C_f$ implies that $f(w) = 1$. However, since $w \in C_f$, we also have that $f(w') = 1$.

Now suppose that $f(u) = 1$. We show that we may assume that $f(w) = -1$. For suppose $f(w) = 1$. If $f(w') = 1$, then the function $g : V \to \{-1, 1\}$ defined by

$$g(z) = \begin{cases} -1 & \text{if } z = u, \\ f(z) & \text{otherwise} \end{cases}$$

is a signed kSF such that $g(V) = f(V) - 2$, which is a contradiction. Hence, $f(w') = -1$. But then the function $g : V \to \{-1, 1\}$ defined by

$$g(z) = \begin{cases} 1 & \text{if } z = w', \\ -1 & \text{if } z = w, \\ f(z) & \text{otherwise} \end{cases}$$

is a signed kSF such $g(V) = f(V)$, while $V(P_t) - \{v\} \subseteq C_g$, $|\{v \in S \mid f(v) = -1\}| = |\{v \in S \mid g(v) = -1\}|$, $g(w') = 1$ and $g(w) = -1$. We assume, therefore, that $f(w) = -1$. Then $w \in C_f$ implies that $f(w') = 1$. But then $g : V \to \{-1, 1\}$ defined by

$$g(z) = \begin{cases} 1 & \text{if } z = w, \\ -1 & \text{if } z = u, \\ f(z) & \text{otherwise} \end{cases}$$

is a signed kSF such $g(V) = f(V)$, $V(P_t) - \{v\} \subseteq C_g$, $|\{v \in S \mid f(v) = -1\}| = |\{v \in S \mid g(v) = -1\}|$ while $g(w') = g(w) = g(v) = 1$ and $g(u) = -1$. We have found the desired function. It now follows that the restriction of f to $V(P_t) - \{u, v\}$ is a signed dominating function of P_{t-2}, so that $f(V) \geqslant 2(\lfloor \frac{[2(t-2)+4]}{3} \rfloor + 1 + k - t + 1) - p = 2(\lfloor \frac{2}{3}t \rfloor + 1 + k - t + 1) - p = 2(\lfloor \frac{2}{3}t \rfloor - t + k + 2) - p = 2(k - \lceil \frac{t}{3} \rceil + 2) - p$, and the result follows. □

In what follows, denote $C_f \cap S$ by S_c.

Claim 5. *If $v \in C_f$ and $f(u) = 1$, then $|S_c| = \lfloor s/2 \rfloor + r$ for some nonnegative integer r and $\{u, v\} \subseteq C_f$. Moreover, if $r \geqslant 1$, then $k = t + \lfloor s/2 \rfloor + r$.*

Proof. Since $f(u) = f(v) = 1$ and $v \in C_f$, it follows that $|S_c| \geqslant \lfloor \frac{s}{2} \rfloor$ and $\{u, v\} \subseteq C_f$. Hence, there exists a nonnegative integer r such that $|S_c| = \lfloor \frac{s}{2} \rfloor + r$.

Suppose $r \geqslant 1$. Then the proof of Claim 3 shows that $V(P_t) \subseteq C_f$. Hence, $\lfloor \frac{s}{2} \rfloor + r + t = |C_f| \geqslant k$. If $k < t + \lfloor \frac{s}{2} \rfloor + r$, then, if $v' \in S_c$, the function $g : V \to \{-1, 1\}$ defined by

$$g(z) = \begin{cases} -1 & \text{if } z = v', \\ f(z) & \text{otherwise} \end{cases}$$

is a signed kSF such that $g(V) = f(V) - 2$, which is a contradiction. Consequently, $k = t + \lfloor \frac{s}{2} \rfloor + r$. □

Claim 6. *If $v \in C_f$ and $f(u) = -1$, then $|S_c| = \lfloor \frac{s}{2} \rfloor + 1 + r$ for some nonnegative integer r, $u \in C_f$, $|C_f \cap V(P_t)| \geqslant 3$ and $t \geqslant 4$. Moreover, if $r \geqslant 1$, then $k = t + \lfloor \frac{s}{2} \rfloor + 1 + r$.*

Proof. Clearly, $|S_c| = \lfloor \frac{s}{2} \rfloor + 1 + r$ for some nonnegative integer r. Let $\ell = |C_f \cap V(P_t)|$ and let $v' \in S_c$.

We first show that $u \in C_f$. Suppose, to the contrary, that $u \notin C_f$. The function $g : V \to \{-1, 1\}$ defined by

$$g(z) = \begin{cases} 1 & \text{if } z = u, \\ -1 & \text{if } z = v', \\ f(z) & \text{otherwise} \end{cases}$$

is a signed kSF such $g(V) = f(V)$ (since $u \in C_g$ and $g[v] = f[v] \geqslant 1$). However, $|\{v \in S \mid f(v) = -1\}| < |\{v \in S \mid g(v) = -1\}|$, which contradicts our choice of f.

We conclude that $u \in C_f$, so that $\ell \geqslant 2$. We now show that $\ell \geqslant 3$. For suppose, to the contrary, that $\ell = 2$. Since $u \in C_f$ and $f(u) = -1$, u is not an end vertex of P_t and $f(w) = 1$. Also, since $C_f \cap V(P_t) = \{u, v\}$, $w \notin C_f$. It now follows that the function $g : V \to \{-1, 1\}$ defined by

$$g(z) = \begin{cases} 1 & \text{if } z = u, \\ -1 & \text{if } z = v', \\ f(z) & \text{otherwise} \end{cases}$$

is a signed kSF such $g(V) = f(V)$ (since $w \in C_g$ and $g[v] = f[v] \geqslant 1$). However, $|\{v \in S \mid f(v) = -1\}| < |\{v \in S \mid g(v) = -1\}|$, which contradicts our choice of f. This final contradiction shows that $\ell \geqslant 3$.

Consequently, $t \geqslant 3$. If $t = 3$, then $f(u) = -1$ implies that $w \notin C_f$, contradicting the fact that $\ell \geqslant 3$. Hence, $t \geqslant 4$. To show that $k = t + \lfloor \frac{s}{2} \rfloor + 1 + r$ if $r \geqslant 1$ is similar to the proof of Claim 5 and is therefore omitted. □

Claim 7. *If $v \in C_f$, then $f(V) \geqslant 2(k - \lceil \frac{t}{3} \rceil + 1) - p$.*

Proof. Let $\ell = |C_f \cap V(P_t)|$. Then $\ell + |S_c| \geqslant k$. We distinguish between two cases.

Case 1: $f(u) = 1$. In this case the function obtained by restricting f to $V(P_t)$ is an ℓSF for P_t and $1 \leqslant \ell \leqslant t$. Hence,

$$\begin{aligned}
f(V) &\geqslant 2\left(\left\lfloor\frac{2\ell+4}{3}\right\rfloor + |S_c|\right) - p \\
&\geqslant 2\left(\left\lfloor\frac{2(k-|S_c|)+4}{3}\right\rfloor + |S_c|\right) - p \\
&= 2\left(\left\lfloor\frac{2k}{3} - 2\frac{|S_c|}{3} + \frac{4}{3} + |S_c|\right\rfloor\right) - p \\
&= 2\left(\left\lfloor\frac{2k}{3} + \frac{|S_c|}{3} + \frac{4}{3}\right\rfloor\right) - p \\
&\geqslant 2\left(\left\lfloor\frac{2k}{3} + \frac{k}{3} - \frac{t}{3} + \frac{4}{3}\right\rfloor\right) - p \\
&\geqslant 2\left(k - \left\lceil\frac{t}{3}\right\rceil + 1\right) - p.
\end{aligned}$$

Case 2: $f(u) = -1$. By Claim 6, the function obtained by restricting f to $V(P_t) - \{u, v\}$ is an $(\ell - 2)$ SF for $P_t - u - v$ and $1 \leqslant \ell - 2 \leqslant t - 2$. Hence,

$$\begin{aligned}
f(V) &\geqslant 2\left(\left\lfloor\frac{2(\ell-2)+4}{3}\right\rfloor + 1 + |S_c|\right) - p \\
&\geqslant 2\left(\left\lfloor\frac{2(k-|S_c|-2)+4}{3}\right\rfloor + 1 + |S_c|\right) - p \\
&= 2\left(\left\lfloor\frac{2k}{3} - 2\frac{|S_c|}{3} + 1 + |S_c|\right\rfloor\right) - p \\
&= 2\left(\left\lfloor\frac{2k}{3} + \frac{|S_c|}{3} + 1\right\rfloor\right) - p \\
&\geqslant 2\left(\left\lfloor\frac{2k}{3} + \frac{k}{3} - \frac{t}{3} + 1\right\rfloor\right) - p \\
&= 2\left(k - \left\lceil\frac{t}{3}\right\rceil + 1\right) - p. \quad \square
\end{aligned}$$

We now show that if $k \leqslant t + \lfloor\frac{s}{2}\rfloor - 2$ and $t \equiv 0 \pmod 3$ or $k \leqslant t + \lfloor\frac{s}{2}\rfloor$ and $t \equiv 1 \pmod 3$ or $k \leqslant t + \lfloor\frac{s}{2}\rfloor - 1$ and $t \equiv 2 \pmod 3$, then $f(V) \geqslant 2(k - \lceil\frac{t}{3}\rceil + 2) - p$.

If $v \notin C_f$, then Claim 4 implies that $f(V) \geqslant 2(k - \lceil \frac{t}{3} \rceil + 2) - p$, and we are done. We assume therefore that $v \in C_f$. Let

$$r' = \begin{cases} 2 & \text{if } t \equiv 0 \pmod 3, \\ 0 & \text{if } t \equiv 1 \pmod 3, \\ 1 & \text{if } t \equiv 2 \pmod 3) \end{cases}$$

and notice that $k \leqslant t + \lfloor \frac{s}{2} \rfloor - r'$.

We first consider the case when $f(u) = 1$. Claim 5 implies that $|S_c| = \lfloor \frac{s}{2} \rfloor + r$, where r is a nonnegative integer. If $r \geqslant 1$, then $k = t + \lfloor \frac{s}{2} \rfloor + r$, contradicting our assumption that $k \leqslant t + \lfloor \frac{s}{2} \rfloor$. Consequently, $r = 0$. Let $\ell = |C_f \cap V(P_t)|$. Then

$$\ell + \left\lfloor \frac{s}{2} \right\rfloor = |C_f| \geqslant k \quad \text{and} \quad f(V) \geqslant 2\left(\left\lfloor \frac{2\ell + 4}{3} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor\right) - p$$

$$= 2\left(\left\lfloor \frac{2(k - \lfloor \frac{s}{2} \rfloor) + 4}{3} + \left\lfloor \frac{s}{2} \right\rfloor \right\rfloor\right) - p$$

$$= 2\left(\left\lfloor \frac{2k}{3} + \frac{\lfloor \frac{s}{2} \rfloor}{3} + \frac{4}{3} \right\rfloor\right) - p$$

$$\geqslant 2\left(\left\lfloor \frac{2k}{3} + \frac{k}{3} - \frac{t}{3} + \frac{r' + 4}{3} \right\rfloor\right) - p$$

$$= 2(k - \left\lceil \frac{t}{3} \right\rceil + 2) - p.$$

Next, consider the case when $f(u) = -1$. Claim 6 implies that $|S_c| = \lfloor \frac{s}{2} \rfloor + 1 + r$, where r is a nonnegative integer. If $r \geqslant 1$, then $k = t + \lfloor \frac{s}{2} \rfloor + 1 + r$, contradicting our assumption that $k \leqslant t + \lfloor \frac{s}{2} \rfloor$. Consequently, $r = 0$. Let $\ell = |C_f \cap V(P_t)|$. Then Claim 6 implies that $u \in C_f$, $3 \leqslant \ell \leqslant t$ and $t \geqslant 4$. Furthermore, $\ell + \lfloor \frac{s}{2} \rfloor + 1 = |C_f| \geqslant k$, while f restricted to $V(P_t) - \{u, v\}$ is an $(\ell - 2)$SF for $P_t - u - v$. Hence,

$$f(V) \geqslant 2\left(\left\lfloor \frac{2(\ell - 2) + 4}{3} \right\rfloor + 1 + \left\lfloor \frac{s}{2} \right\rfloor + 1\right) - p$$

$$= 2\left(\left\lfloor \frac{2\ell}{3} \right\rfloor + 1 + \left\lfloor \frac{s}{2} \right\rfloor + 1\right) - p$$

$$\geqslant 2\left(\left\lfloor \frac{2k}{3} - 2\frac{(\lfloor \frac{s}{2} \rfloor + 1)}{3} \right\rfloor + 1 + \left\lfloor \frac{s}{2} \right\rfloor + 1\right) - p$$

$$= 2\left(\left\lfloor \frac{2k}{3} + \frac{\lfloor \frac{s}{2} \rfloor + 1}{3} + 1 \right\rfloor\right) - p$$

$$\geqslant 2\left(\left\lfloor \frac{2k}{3} + \frac{k-t+r'+1}{3} + 1 \right\rfloor\right) - p$$

$$= 2\left(\left\lfloor k - \frac{t}{3} + \frac{r'+4}{3} \right\rfloor\right) - p$$

$$= 2\left(k - \left\lceil \frac{t}{3} \right\rceil + 2\right) - p.$$

Subsequently, we show that the value of $2(k - \lceil \frac{t}{3} \rceil + 2) - p$ can be attained if $k \leq t + \lfloor \frac{s}{2} \rfloor$. If $t = 2$, define $f : V(P_t) \to \{-1, 1\}$ by $f(u) = f(v) = 1$ and extend f to a kSF of G by assigning the value 1 to $k-t+1$ vertices of S (which is possible since $k \leqslant t+s-1$) and -1 to the remaining vertices of S. Then $f(V) = 2(2+(k-1)) - p = 2(k-1+2) - p = 2(k - \lceil \frac{t}{3} \rceil + 2) - p$. If $t = 3$, define $f : V(P_t) \to \{-1, 1\}$ by $f(w) = f(u) = f(v) = 1$ and extend f to a kSF of G by assigning the value 1 to $k-t+1$ vertices of S (which is possible since $k \leqslant t+s-1$) and -1 to the remaining vertices of S. Then $f(V) = 2(3+(k-2)) - p = 2(k-1+2) - p = 2(k - \lceil \frac{t}{3} \rceil + 2) - p$. If $t \geqslant 4$, take a minimum signed dominating function on $P_t - u - v$ and extend it to a function $f : V(G) \to \{-1, 0, 1\}$ by assigning the value 1 to v and to $k-t+1$ vertices of S (which is possible since $k \leqslant t+s-1$) and -1 to the remaining vertices of S and to the vertex u. Then $|C_f| \geqslant (k-t+1) + (t-1) = k$, while f has weight

$$2\left(\left\lfloor \frac{2(t-2)+4}{3} \right\rfloor + 1 + k - t + 1\right) - p = 2\left(\left\lfloor \frac{2t}{3} + 1 + k - t + 1 \right\rfloor\right) - p$$

$$= 2\left(\left\lfloor k - \frac{t}{3} + 2 \right\rfloor\right) - p$$

$$= 2\left(k - \left\lceil \frac{t}{3} \right\rceil + 2\right) - p.$$

To complete the proof we show that if $k \geqslant t + \lfloor \frac{s}{2} \rfloor - 1$ and $t \equiv 0 \pmod 3$ or $k \geqslant t + \lfloor \frac{s}{2} \rfloor + 1$ and $t \equiv 1 \pmod 3$ or $k \geqslant t + \lfloor \frac{s}{2} \rfloor$ and $t \equiv 2 \pmod 3$, then $\gamma_{\mathrm{ks}}^{-11}(G) = 2(k - \lceil \frac{t}{3} \rceil + 1) - p$.

Let $V(P_t) = \{v_1, \ldots, v_t\}$ where $v_t = v$. First, consider the case when $t \equiv 0 \pmod 3$. If $k = t + \lfloor \frac{s}{2} \rfloor - 1$, define $f : V(P_t) \to \{-1, 1\}$ by

$$f(v_i) = \begin{cases} -1 & \text{if } i \equiv 1 \pmod 3, \\ 1 & \text{otherwise} \end{cases}$$

and extend f to a kSF of G by assigning 1 to $\lfloor \frac{s}{2} \rfloor$ vertices of S and -1 to the remaining vertices of S. Then $f(V) = 2(\lfloor \frac{s}{2} \rfloor + 2\frac{t}{3}) - p = 2(k-t+1+2\frac{t}{3}) - p = 2(k - \lceil \frac{t}{3} \rceil + 1) - p$.

If $k = t + \lfloor \frac{s}{2} \rfloor + r$ for some integer $r \geqslant 0$, define $f : V(P_t) \to \{-1, 1\}$ by

$$f(v_i) = \begin{cases} -1 & \text{if } i \equiv 1 \pmod 3 \text{ and } i \geqslant 4, \\ 1 & \text{otherwise} \end{cases}$$

and extend f to a kSF of G by assigning 1 to $\lfloor\frac{s}{2}\rfloor+r$ vertices of S and -1 to the remaining vertices of S. Then $f(V)=2(\lfloor\frac{s}{2}\rfloor+r+2\frac{t-3}{3}+3)-p=2(k-t+2\frac{t}{3}+1)-p=2(k-\lceil\frac{t}{3}\rceil+1)-p$.

Next, consider the case when $t\equiv 1\,(\mathrm{mod}\,3)$ and $k=t+\lfloor\frac{s}{2}\rfloor+r$ for some integer $r\geqslant 1$. Define $f:V(P_t)\to\{-1,1\}$ by

$$f(v_i)=\begin{cases}-1 & \text{if } i\equiv 0\,(\mathrm{mod}\,3),\\ 1 & \text{otherwise}\end{cases}$$

and extend f to a kSF of G by assigning 1 to $\lfloor\frac{s}{2}\rfloor+r$ vertices of S and -1 to the remaining vertices of S. Then $f(V)=2(\lfloor\frac{s}{2}\rfloor+r+2\frac{t-1}{3}+1)-p=2(k-t+2\frac{t-1}{3}+1)-p=2(k-\lceil\frac{t}{3}\rceil+1)-p$.

Finally consider the case when $t\equiv 2\,(\mathrm{mod}\,3)$ and $k=t+\lfloor\frac{s}{2}\rfloor+r$ for some integer $r\geqslant 0$. Define $f:V(P_t)\to\{-1,1\}$ by

$$f(v_i)=\begin{cases}-1 & \text{if } i\equiv 0\,(\mathrm{mod}\,3),\\ 1 & \text{otherwise}\end{cases}$$

and extend f to a kSF of G by assigning 1 to $\lfloor\frac{s}{2}\rfloor+r$ vertices of S and -1 to the remaining vertices of S. Then $f(V)=2(\lfloor\frac{s}{2}\rfloor+r+2\frac{t-2}{3}+2)-p=2(k-t+2\frac{t-2}{3}+2)-p=2(k-\lceil\frac{t}{3}\rceil+1)-p$. $\square$

3. The minus k-subdomination number of comets

In this section we compute the value of $\gamma_{\mathrm{ks}}^{-101}$ for comets. The following three results will prove to be useful.

In order to state the first result we need the following definitions. The kSF f is called *minimal* if no $g<f$ is a kSF. Let f be a minus kSF for the graph $G=(V,E)$. We use three sets for such an f:

$$B_f=\{v\in V\,|\,f[v]=1\},$$
$$P_f=\{v\in V\,|\,f(v)\geqslant 0\}$$

and

$$C_f=\{v\in V\,|\,f[v]\geqslant 1\}.$$

For $A,B\subseteq V$, we say that A *dominates* B (denoted by $A\succ B$) if for each $b\in B$ we have $N[b]\cap A\neq\emptyset$.

The following results are due to Broere et al. [2].

Theorem 8. *A minus kSF f is minimal if and only if for each k-subset K of C_f we have $B_f\cap K\succ P_f$.*

Theorem 9. *If $p\geqslant 2$ is an integer and $1\leqslant k\leqslant p-1$, then $\gamma_{\mathrm{ks}}^{-101}(P_p)=\lceil\frac{k}{3}\rceil+k-p+1$.*

Hattingh and Ungerer [7] established the following result.

Theorem 10. *If T is a tree of order $p \geqslant 2$ and k is an integer such that $1 \leqslant k \leqslant p-1$, then*

$$\gamma_{\mathrm{ks}}^{-101}(T) \geqslant k - p + 2.$$

We are now ready to compute $\gamma_{\mathrm{ks}}^{-101}$ for comets.

Theorem 11. *Let p, s and t be positive integers such that $p = s + t$, let k be an integer such that $1 \leqslant k \leqslant p - 1$ and let $G = C_{s,t}$. If $t \geqslant 2$ and $s \geqslant 2$, then*

$$\gamma_{\mathrm{ks}}^{-101}(G) = \begin{cases} k - p + 2 & \text{if } 1 \leqslant k \leqslant s, \\ \lceil \frac{k-s+1}{3} \rceil + k - p + 1 & \text{if } s + 1 \leqslant k \leqslant p. \end{cases}$$

Proof. Since G is a tree, Theorem 10 implies that $\gamma_{\mathrm{ks}}^{-101}(G) \geqslant k - p + 2$. We now show that if $1 \leqslant k \leqslant s$, then $\gamma_{\mathrm{ks}}^{-101}(G) \leqslant k - p + 2$, while if $s + 1 \leqslant k \leqslant p$, then $\gamma_{\mathrm{ks}}^{-101}(G) \leqslant \lceil (k - s + 1)/3 \rceil + k - p + 1$.

Suppose first that $1 \leqslant k \leqslant s$. Let S denote the set of end vertices of the star and let $S' \subseteq S$ be any subset such that $|S'| = k$. Define $f : V \to \{-1, 0, 1\}$ by

$$f(v) = \begin{cases} 1 & \text{if } v = v_1, \\ 0 & \text{if } v \in S', \\ -1 & \text{otherwise.} \end{cases}$$

Then $S' \subseteq C_f$ with $|S'| = k$, while $f(V(G)) = k0 + (s - k)(-1) + 1 + (t - 1)(-1) = k - s + 1 + 1 - t = k - p + 2$. Hence, $\gamma_{\mathrm{ks}}^{-101}(G) \leqslant k - p + 2$.

Suppose next that $s + 1 \leqslant k \leqslant p$. Let $\{v_1, \ldots, v_{t+1}\}$ denote the vertex set of the path P_{t+1} and identify the centre of the star $K_{1,s-1}$ with the vertex v_2 of P_{t+1}. Note that the graph G obtained in this way is isomorphic to G. Let S denote the end vertices of the star $K_{1,s-1}$. Define $f : V(G) \to \{-1, 0, 1\}$ by

$$f(v) = \begin{cases} 1 & \text{if } v = v_i \text{ where } i \equiv 2 \,(\mathrm{mod}\, 3) \text{ and } i \leqslant k - s + 2 \\ 0 & \text{if } v = v_i \text{ where } i \equiv 0 \text{ or } 1 \,(\mathrm{mod}\, 3) \text{ and } i \leqslant k - s + 2 \text{ or } v \in S \\ -1 & \text{otherwise.} \end{cases}$$

Then it is easily verified that $\{v_1, \ldots, v_{k-s+1}\} \cup S \subseteq C_f$ and that $f(V(G)) = \lceil (k - s + 1)/3 \rceil + k - p + 1$. Hence, f is a minus kSF for G, so that $\gamma_{\mathrm{ks}}^{-101}(G) \leqslant \lceil (k - s + 1)/3 \rceil + k - p + 1$.

All that remains to be shown is that $\gamma_{\mathrm{ks}}^{-101}(G) \geqslant \lceil (k - s + 1)/3 \rceil + k - p + 1$. Let S denote the set of end vertices of the star in G and let $V = V(G)$. Among all minimum kSF's for G, let f be chosen as to maximise the number of vertices in S that are assigned a 0 by f.

We complete the proof by considering three cases.

Case 1: $f(v_1) = -1$. Then $v \notin C_f$ for all $v \in S$. If $f(v) \geqslant 0$ for some $v \in S$, then, by the minimality of f (cf. Theorem 8), $f[v_1] = 1$. We conclude that there exists $u \in S$ such that $f(u) = 1$. Then $g : V \to \{-1, 0, 1\}$ defined by

$$g(z) = \begin{cases} 0 & \text{if } z \in \{u, v_1\}, \\ f(z) & \text{otherwise} \end{cases}$$

is a minimum minus kSF for G which assigns more zeros to the vertices of S than f, which contradicts the choice of f. We conclude that $f(v) = -1$ for all $v \in S$ and that f restricted to $V(P_t)$ is a kSF for P_t. Using Theorem 9, we conclude that $f(V) \geqslant \gamma_{\mathrm{ks}}^{-101}(P_t) + s(-1) = \lceil \frac{k}{3} \rceil + k - t + 1 - s \geqslant \lceil (k - r + 1)/3 \rceil + k - p + 1$.

Case 2: $f(v_1) = 0$. If $f(v) = 1$ for some $v \in S$, then $g : V \to \{-1, 0, 1\}$ defined by

$$g(z) = \begin{cases} 0 & \text{if } z = v, \\ 1 & \text{if } z = v_1, \\ f(z) & \text{otherwise} \end{cases}$$

is a minimum minus kSF for G which assigns more zeros to the vertices of S than f, which contradicts the choice of f. We conclude that $f(v) \leqslant 0$ for all $v \in S$ and that f restricted to $V(P_t)$ is a kSF for P_t. It now follows that $f(V) \geqslant \gamma_{\mathrm{ks}}^{-101}(P_t) + s(-1) = \lceil \frac{k}{3} \rceil + k - t + 1 - s \geqslant \lceil (k - r + 1)/3 \rceil + k - p + 1$.

Case 3: $f(v_1) = 1$.

Case 3.1: There exists $v \in S$ such that $f(v) = -1$. If $f(u) = 1$ for some $u \in S - \{v\}$, then $g : V \to \{-1, 0, 1\}$ defined by

$$g(z) = \begin{cases} 0 & \text{if } z \in \{u, v\}, \\ f(z) & \text{otherwise} \end{cases}$$

is a minimum minus kSF for G which assigns more zeros to vertices of S than f, which contradicts the choice of f. We conclude that $f(v) \leqslant 0$ for all $v \in S$, all vertices in $S \cap C_f$ are assigned a 0 by f and all vertices in $S \cap \overline{C}_f$ are assigned a -1 by f. Let $|S \cap C_f| = n$. Since $v \notin C_f$, it follows that $n < s$. Furthermore, $k - s \geqslant 1$ implies that $k - n \geqslant 1$. Since f restricted to $V(P_t)$ is a $(k - n)$SF for P_t, we have

$$\begin{aligned} f(V) &\geqslant \left\lceil \frac{k-n}{3} \right\rceil + (k - n) - t + 1 + (-1)(s - n) \\ &= \left\lceil \frac{k-n}{3} \right\rceil + k - p + 1 \\ &\geqslant \left\lceil \frac{k-s+1}{3} \right\rceil + k - p + 1. \end{aligned}$$

Case 3.2: $f(v) \geqslant 0$ for all $v \in S$. If $f(v) = 1$ for some $v \in S$, then, by the minimality of f, $f[v_1] = 1$. Since $f(u) \geqslant 0$ for all $u \in S - \{v\}$, we must have that $f(v_2) = -1$. Then $g : V \to \{-1, 0, 1\}$ defined by

$$g(z) = \begin{cases} 0 & \text{if } z \in \{v, v_2\}, \\ f(z) & \text{otherwise} \end{cases}$$

is a minimum minus kSF for G which assigns more zeros to vertices of S than f, which contradicts the choice of f. We conclude that $f(v) = 0$ for all $v \in S$. Let $v \in S$ and let P' be the path induced by the vertices $\{v, v_1, \ldots, v_t\}$. Then f restricted to $V(P')$ is a $(k-s+1)$SF for P', so that $f(V) \geqslant \gamma_{ks}^{-101}(P') + f(S - \{v\}) = \lceil (k-s+1)/3 \rceil + (k-s+1) - (t+1) + 1 = \lceil (k-s+1)/3 \rceil + k - p + 1$.

Hence, in all cases, $f(V) \geqslant \lceil (k-s+1)/3 \rceil + k - p + 1$ and the result follows. □

Note, in closing, that $\gamma^{-}(G_{s,t}) = \lceil (t+1)/3 \rceil$, where s and t are positive integers.

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References

[1] L.W. Beineke, M.A. Henning, Opinion functions on trees, Discrete Math., to appear.
[2] I. Broere, J.E. Dunbar, J.H. Hattingh, Minus k-subdomination in graphs, Ars Combin., to appear.
[3] I. Broere, J.H. Hattingh, M.A. Henning, A.A. McRae, Majority domination in graphs, Discrete Math. 138 (1995) 125–135.
[4] E.J. Cockayne, C.M. Mynhardt, On a generalisation of signed dominating functions of graphs. Ars Combin. 43 (1996) 235–245.
[5] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning, A.A. McRae, Minus domination in graphs, Comput. Math. Appl., submitted.
[6] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning, P.J. Slater, Signed domination in graphs, Proc. 7th Internat. Conf. in Graph Theory, Combinatorics, Algorithms and Applications (Wiley, 1995) 311–322.
[7] J.H. Hattingh, E. Ungerer, Minus k-subdomination in graphs II, Discrete Math., to appear.